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The random Ising model on cacti lattices

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Abstract. We study the random bond Ising model on pseudolattices containing triangles. By using the star-triangle transformation, the problem can be mapped onto a Bethe lattice where the solution to the random bond problem is well known. We look at the solution for the dilute ferromagnet in detail. It is shown that the transition temperature goes to zero at the percolation concentration as we would expect.

1. Introduction

The study of the thermodynamics of random systems is difficult and it may be useful to study idealised and even unphysical systems. In this paper we study the random bond Ising model on a class of pseudolattices containing triangles. We show that the model can be solved exactly and look at the solution in detail for the special case of the dilute ferromagnet.

The Hamiltonian for the system is

$$H = - \sum_{\langle ij \rangle} J_{ij} \sigma_i \sigma_j \quad (1)$$

where $\sigma_i = \pm 1$ and the summation goes over only nearest neighbours such that each bond is counted once. The exchange interactions J_{ij} are distributed at random according to a probability distribution function $P(J_{ij})$ which is the same for all bonds in the lattice. That is the probability distribution function for the whole system factorises into a product of single bond probabilities $P(J_{ij})$. We are therefore dealing with a quenched random system (Brout 1959, Mazo 1963, Thorpe and Beeman 1976). The free energy is calculated by generating a particular distribution of bonds. If the system is thermodynamically large ($N \rightarrow \infty$) it is not necessary to average over all equivalent systems as all possible configurations will be present in a single member of the ensemble. However it may be convenient to do so and we will adopt that point of view in the next section where we find the free energy of a Bethe lattice that can be divided into two sublattices, one of which has coordination 3 and the other z . The random Bethe lattice is very simple to treat because the precise arrangement of bonds is not important and the quenched and annealed problems give the same phase diagram provided that all the interactions have the same sign (Matsubara 1974, Thorpe and Beeman 1976, Young 1976). This solution is briefly derived in the next section.

In § 3, we utilise the asymmetric star-triangle transformation (see for example Syozi 1972) and generate the solution for a certain class of cacti lattices. This solution

is studied in detail in § 4 for the dilute ferromagnet where the phase diagram is given for various cases. It is shown that the transition temperature goes to zero at the bond percolation concentration which is derived independently in the appendix.

2. Bethe lattice

In figure 1, we show a Bethe lattice with site coordination numbers 3 and z alternately ($z = 2$ for the case drawn in the diagram). The simplifying feature of the Bethe lattice is that there is a unique path connecting any two points. The Hamiltonian (1) can be

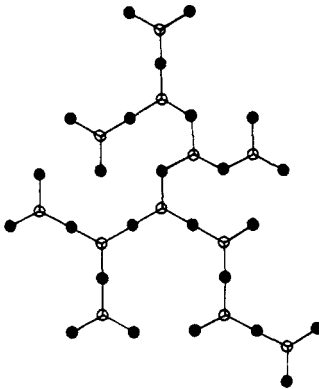


Figure 1. A Bethe lattice that has two sublattices with coordination number 3 (open circles) and $z = 2$ (full circles).

very simply solved by noting that we can transform to new variables τ_{ij} such that

$$\tau_{ij} = \sigma_i \sigma_j \tag{2}$$

where the $\tau_{ij} = \pm 1$ are *independent* variables for each bond. Therefore the Hamiltonian (1) becomes a sum of *single-particle Hamiltonians* and the thermodynamic functions of interest can be found trivially. For example, the energy E is given by

$$E = - \sum_{\langle ij \rangle} J_{ij} \tanh(\beta J_{ij}) = - \int J \tanh(\beta J) P(J) dJ. \tag{3}$$

The susceptibility per site χ can be written as

$$\chi = \frac{\beta}{N} \sum_{i,k} \langle \sigma_i \sigma_k \rangle \tag{4}$$

where N is the total number of sites and the summation goes over all sites. Because of the property of the Bethe lattice that there is a unique path between any two points, we may write

$$\begin{aligned} \langle \sigma_i \sigma_k \rangle &= \langle \sigma_i \sigma_{i+1} \sigma_{i+1} \sigma_{i+2} \dots \sigma_{k-1} \sigma_k \rangle \\ &= \langle \tau_{i,i+1} \tau_{i+1,i+2} \dots \tau_{k-1,k} \rangle \\ &= \langle \tau_{i,i+1} \rangle \langle \tau_{i+1,i+2} \rangle \dots \langle \tau_{k-1,k} \rangle, \end{aligned} \tag{5}$$

therefore

$$\chi = \beta(1 + \epsilon)[1 - \epsilon + \frac{1}{2}(z + 3)\epsilon]/[1 - 2(z - 1)\epsilon^2] \tag{6}$$

where

$$\epsilon = \int \tanh(\beta J)P(J) dJ \tag{7}$$

is the average value of the nearest-neighbour spin-spin correlation function.

The susceptibility has this simple form because of the observation (5) that turns the susceptibility into the sum of terms in a geometric series. The phase transition occurs when the susceptibility diverges, i.e.

$$\epsilon_c = \int \tanh(\beta_c J)P(J) dJ = [2(z - 1)]^{-1/2} \tag{8}$$

where the subscript c denotes the value at the critical point. Notice that for a *dilute ferromagnet* the condition in (8) reduces to

$$\epsilon_c = p \tanh(\beta_c J) = [2(z - 1)]^{-1/2} \tag{9}$$

where p is the concentration of ferromagnetic bonds. The transition temperature goes to zero at p_c given by

$$p_c = \frac{1}{[2(z - 1)]^{1/2}}. \tag{10}$$

This is just the percolation concentration for this Bethe lattice as shown in the appendix (equation (A.11)).

These results are derived for a Bethe lattice with sites that have coordination alternately 3 and z . They are a **simple** extension of known results for the Bethe lattice with a single kind of site. We are restricting our considerations in this and subsequent sections to the case where all the bonds are ferromagnetic, i.e. $J_{ij} \geq 0$, so that we can just consider the *ferromagnetic* susceptibility (4). In cases where there are interactions of both signs, the situation is rather more complex and other susceptibilities may diverge at the phase transition.

3. Cacti lattices

The cacti lattices for which we can solve the random bond Ising model are shown in figure 2. They are formed by constructing triangles around all the sites in the Bethe lattice with coordination 3 so that we have a lattice made up of triangles such that z triangles come together at their vertices and each site has $2z$ nearest neighbours. The only rings in these lattices are the triangles themselves; there are no rings of triangles (for a further discussion see Domb 1960, p 284).

Mathematically the transformation to the cacti lattices from the Bethe lattice can be achieved via the star-triangle transformation first used by Onsager (1944) and reviewed in detail for the asymmetric case by Syozi (1972) whose treatment we follow here. In figure 3, we show a star with interaction parameters (βJ) denoted by K_1 , K_2 and K_3 and a triangle with interaction parameters L_1 , L_2 and L_3 . By doing the trace

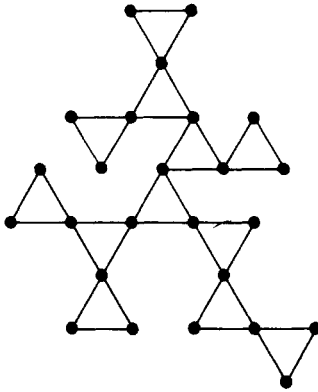


Figure 2. A triangular cacti lattice formed from the Bethe lattice in figure 1 by constructing triangles joining the three neighbours of the sites represented by open circles. The case $z = 2$ is drawn.

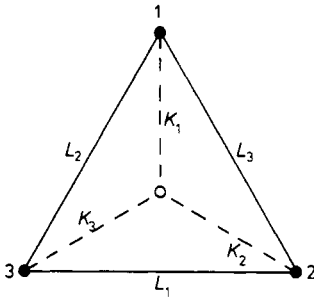


Figure 3. The figure illustrates the parameters used in the star-triangle transformation discussed in § 3.

over the spin σ_0 , we have the mapping

$$\sum_{\sigma_0 = \pm 1} e^{K_1 \sigma_0 \sigma_1} e^{K_2 \sigma_0 \sigma_2} e^{K_3 \sigma_0 \sigma_3} = A e^{L_1 \sigma_2 \sigma_3} e^{L_2 \sigma_3 \sigma_1} e^{L_3 \sigma_1 \sigma_2} \tag{11}$$

i.e.

$$2 \cosh(K_1 \sigma_1 + K_2 \sigma_2 + K_3 \sigma_3) = A e^{L_1 \sigma_2 \sigma_3} e^{L_2 \sigma_3 \sigma_1} e^{L_3 \sigma_1 \sigma_2}. \tag{12}$$

This is an operator identity, where A is a constant. If we put all the spins up, we get

$$2 \cosh(K_1 + K_2 + K_3) = A e^{L_1} e^{L_2} e^{L_3} \tag{13}$$

and if we have spin-1 down and the other two up,

$$2 \cosh(-K_1 + K_2 + K_3) = A e^{L_1} e^{-L_2} e^{-L_3}. \tag{14}$$

From (13) and (14) we can eliminate A

$$e^{-2L_2} e^{-2L_3} = [\cosh(K_1 - K_2 - K_3)] / [\cosh(K_1 + K_2 + K_3)] \tag{15}$$

and of course two other equations like (15) obtained by permuting the indexes 1, 2, 3.

With a little manipulation, we can get a more useful equation than (15)

$$(1 + e^{-2(L_1+L_2)} + e^{-2(L_2+L_3)} + e^{-2(L_3+L_1)}) \times (1 + \tanh K_1 \tanh K_2 + \tanh K_2 \tanh K_3 + \tanh K_3 \tanh K_1) = 4. \quad (16)$$

This mapping allows all the thermodynamic functions for the cacti lattices to be found. We will concentrate on the phase boundary. Equation (8) may be rewritten

$$1 + 3 \left(\int \tanh(\beta_c J) P(J) dJ \right)^2 = 1 + \frac{3}{2(z-1)} \quad (17)$$

or

$$\begin{aligned} & \iiint [1 + \tanh(\beta_c J_1) \tanh(\beta_c J_2) + \tanh(\beta_c J_2) \tanh(\beta_c J_3) \\ & \quad + \tanh(\beta_c J_3) \tanh(\beta_c J_1)] P(J_1) P(J_2) P(J_3) dJ_1 dJ_2 dJ_3 \\ & = 1 + \frac{3}{2(z-1)}. \end{aligned} \quad (18)$$

Now using the transformation (16) this can be rewritten (noting that $\beta J_i = K_i, i = 1, 2, 3$):

$$\iiint \frac{P(J_1) P(J_2) P(J_3) dJ_1 dJ_2 dJ_3}{1 + e^{-2\beta_c(J_1+J_2)} + e^{-2\beta_c(J_2+J_3)} + e^{-2\beta_c(J_3+J_1)}} = \frac{1}{4} \left(1 + \frac{3}{2(z-1)} \right) = \frac{2z+1}{8(z-1)}. \quad (19)$$

We have written $L_i = \beta J_i, i = 1, 2, 3$ where the J_1 refer to the exchange interactions on the cacti lattice. This equation (19) only involves the probability distribution $P(J)$ and the exchange interactions J on the *cacti lattice* and we need not consider the Bethe lattice from which it was derived any further. Equation (19) is our central result.

In the limit when there is no disorder

$$P(J_1) = \delta(J_1 - J) \quad (20)$$

and (19) gives the transition temperature for the perfect ferromagnetic cactus:

$$e^{-4\beta_c J} = (2z - 3)/(2z + 1) \quad (21)$$

which for $z = 2, 3, 4$ and 6 gives $kT_c/J = 2.4853, 4.7209, 6.8052$ and 10.8777 respectively.

4. The dilute ferromagnet

Equation (19) gives the phase boundary for any distribution $P(J)$. However we must remember that this was derived from the expression for the divergence of the *ferromagnetic* susceptibility on the Bethe lattice. Therefore it only holds if $J \geq 0$ for *all* the bonds. We will examine in more detail the interesting case of the dilute ferromagnet with a concentration p of ferromagnetic bonds:

$$P(J_1) = (1 - p)\delta(J_1) + p\delta(J_1 - J) \quad (22)$$

for which equation (19) becomes

$$\frac{p^3}{1 + 3x^2} + \frac{3p^2(1-p)}{1 + 2x + x^2} + \frac{3p(1-p)^2}{2 + 2x} + \frac{(1-p)^3}{4} = \frac{2z + 1}{8(z - 1)} \quad (23)$$

where $x = e^{-2\beta_c J}$. The phase boundary can easily be found by solving this cubic equation and is shown in figure 4. It can be seen that as $p \rightarrow 1$ we recover the result for the pure ferromagnet and that $T_c \rightarrow 0$ as $x \rightarrow 0$ and we get the equation

$$p_c^3 + 3p_c^2(1-p_c) + \frac{3p_c(1-p_c)^2}{2} + \frac{(1-p_c)^3}{4} = \frac{2z+1}{8(z-1)}$$

i.e.

$$2(z-1)(p_c + p_c^2 - p_c^3) = 1. \quad (24)$$

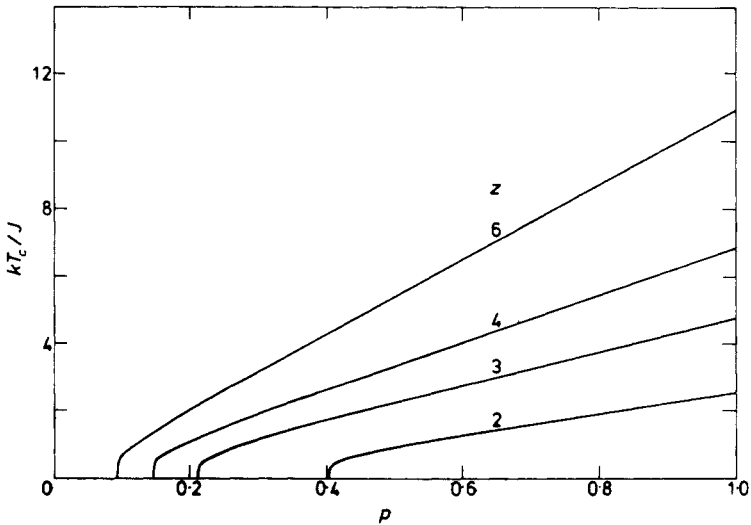


Figure 4. The phase boundaries for the dilute ferromagnet on the triangular cacti lattices with $z = 2, 3, 4$ and 6 .

This of course is just the percolation concentration for the lattice as can be demonstrated directly (see appendix, equation (A.15)). For $z = 2, 3, 4$ and 6 this leads to $p_c = 0.4030, 0.2140, 0.1480$ and 0.0923 . The phase boundary comes into the percolation point with a vertical tangent and moreover all the higher derivatives are infinite also. This also happens in the Bethe lattice (see equation (9)) and is to be expected for any pseudolattice.†

5. Conclusions

We have shown that by using the asymmetric star-triangle transformation on the random bond Bethe lattice, the random bond triangular cacti lattice can be solved. We are currently trying to use this idea to develop an approximate method of calculating the phase boundary in real lattices containing triangles.

† Note added in proof. It has recently been shown that this is true for all lattices (Bergstresser T 1977 *J. Phys. C: Solid St. Phys.* **10** 3831-50).

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Appendix

The percolation concentration for pseudolattices is easily found (see for example Essam 1972). We will calculate it for the cases of interest in this paper as they do not appear to be in the literature. If p is the probability that a bond is occupied and $q = 1 - p$ that it is empty, then we define a probability generating function $P(p, q)$ such that

$$P(p, q) = \sum_{r,s} C_{rs} p^r q^s \tag{A.1}$$

where C_{rs} is the probability of a given bond belonging to a cluster with r occupied bonds surrounded by s empty bonds that isolate it from the rest of the medium. Note that p, q are treated as independent variables in $P(p, q)$. The mean cluster size m (below the percolation concentration) is given by

$$m = \sum_{r,s} C_{rs} r p^r q^s = p \frac{\partial P(p, q)}{\partial p} \tag{A.2}$$

It is useful to introduce an auxiliary function $\Sigma(p, q)$ which sums the probabilities of all bonds coming into a particular given bond. With two kinds of sites we have $\Sigma_1(p, q)$ corresponding to the coordination number 3, and $\Sigma_2(p, q)$ corresponding to coordination number z . We have

$$\Sigma_1 = q^2 + 2pq\Sigma_2 + p^2\Sigma_2^2 \tag{A.3}$$

i.e.

$$\Sigma_1 = (q + p\Sigma_2)^2 \tag{A.4}$$

Similarly

$$\Sigma_2 = (q + p\Sigma_1)^{z-1} \tag{A.5}$$

Clearly when we put $q = 1 - p$, we have $\Sigma_1 = \Sigma_2 = 1$ as solutions to (A.4) and (A.5). We can now write

$$P = q + p\Sigma_1\Sigma_2 \tag{A.6}$$

and so using (A.2)

$$m = p \left(\Sigma_1\Sigma_2 + p\Sigma_1 \frac{\partial \Sigma_2}{\partial p} + p \frac{\partial \Sigma_1}{\partial p} \Sigma_2 \right) \tag{A.7}$$

From (A.4) we have

$$\frac{\partial \Sigma_1}{\partial p} = 2(q + p\Sigma_2) \left(\Sigma_2 + p \frac{\partial \Sigma_2}{\partial p} \right) \tag{A.8}$$

and from (A.5)

$$\frac{\partial \Sigma_2}{\partial p} = (z-1)(q + p\Sigma_1)^{z-1} \left(\Sigma_1 + p \frac{\partial \Sigma_1}{\partial p} \right). \quad (\text{A.9})$$

Putting $q = 1 - p$ into (A.8) and (A.9), we can solve for $\partial \Sigma_1 / \partial p$ and $\partial \Sigma_2 / \partial p$ and hence find that the mean cluster size is given by

$$m = \frac{p(1+2p)[1+(z-1)p]}{1-2(z-1)p^2} \quad (\text{A.10})$$

which diverges at the percolation concentration p_c given by

$$p_c = \frac{1}{[2(z-1)]^{1/2}}. \quad (\text{A.11})$$

Notice from (A.7) that the mean cluster size diverges when $\partial \Sigma_1 / \partial p$ and $\partial \Sigma_2 / \partial p$ diverge.

For the triangular cacti shown in figure 2, it is only necessary to use a single auxiliary function Σ given by

$$\Sigma = q^2 + 2pq^2\Sigma^{z-1} + (3p^2q + p^3)\Sigma^{2z-2} \quad (\text{A.12})$$

for which $\Sigma = 1$ when $q = 1 - p$. Differentiating we have

$$\frac{\partial \Sigma}{\partial p} = 2q^2\Sigma^{z-1} + 2pq^2(z-1)\Sigma^{z-2} \frac{\partial \Sigma}{\partial p} + (6pq + 3p^2)\Sigma^{2z-2} + 2(z-1)(3p^2q + p^3)\Sigma^{2z-3} \frac{\partial \Sigma}{\partial p}. \quad (\text{A.13})$$

Putting $q = 1 - p$, we find that

$$\frac{\partial \Sigma}{\partial p} = \frac{2 + 2p - p^2}{1 - 2(z-1)(p + p^2 - p^3)}. \quad (\text{A.14})$$

This diverges at the percolation concentration p_c given by the solution to the cubic equation:

$$1 = 2(z-1)(p_c + p_c^2 - p_c^3). \quad (\text{A.15})$$

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